Commutative Algebra — December 2024

Back-paper Exam — 100 Points

Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) There are a total of **105** points in this paper. You will be awarded a maximum of **100** points.

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1. [20 Points] Define the nilradical $\mathrm{Nil}(R)$ of a ring R. Prove that $\mathrm{Nil}(R)$ is the intersection of all the prime ideals in R.

2. [15 Points] Let R be a PID and I a nonzero ideal in R. Prove that R/I has only finitely many ideals.

3. [20 Points] Let R be a ring and M a finitely generated R-module. Prove that any endomorphism $\phi\colon M \to M$ satisfies an equation of the form $\phi^n + c_{n-1}\phi^{n-1} + \ldots + c_0 = 0$ for suitable $c_i \in R$.

4. [20 Points] For any ring R and any R-modules M, N, P, establish an isomorphism

$$\hom_R(M \otimes_R N, P) \cong \hom_R(M, \hom_R(N, P)).$$

5. [15 Points] Find the Krull dimension of the ring $\mathbb{Z}[\sqrt{2}][t^2, t^3]$ and the transcendence degree of its fraction field over $\mathbb{Q}$.

6. [15 Points] Find all the maximal ideals in the polynomial ring $\mathbb{R}[x, y]$ that contain both $x^2 + 1$ and $y^2 + 1$.